University of Bahrain

College of Information Technology
Department of Computer Science

ITCS253 Discrete Structures II
First Semester 2013/2014

Exam #1 – 60 Minutes

STUDENT NAME	
STUDENT#	
SECTION	

This exam contains 5 pages (including this cover page) and 6 questions. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You *are not allowed* to use books, notes, or mobiles.

Question	Points	Score
1	9	9
2	7	7
3	4	4
4	6	6
5	6	6
6	8	8
Total:	40	40

Good Job

Instructor: Dr. Ali Alsaffar Sections# 1 & 2

(1) Let a function $f : \{x \in \mathbf{R} \mid x \neq 4\} \to \mathbf{R}$ be defined as $f(x) = \frac{1}{x-4} + 1$.

domain   co-domain

(a) **[3 points]** Find the range of f.

$y = \frac{1}{x-4} + 1$

$y = \frac{1}{x-4} + \frac{(x-4)}{(x-4)} = \frac{1+(x-4)}{(x-4)}$ $1+x-4$ $x-3$

$y = \frac{x-3}{x-4}$

$yx - 4y = x - 3$ $y = \frac{1}{x-4} + 1$

$yx - x = -3 + 4y$

3

$x(y-1) = -3 + 4y$ $y - 1 = \frac{1}{x-4}$

$x = \frac{-3+4y}{y-1}$ ✓

$x - 4 = \frac{1}{y-1} \Rightarrow x = \frac{1}{y-1} + 4$

$\therefore$ Range $\{y \in \mathbb{R} \mid y \neq 1\}$ $\{y \in \mathbb{R} \mid y \neq 1\}$

(b) **[3 points]** Prove that f is one-to-one.

$f(a_1) = f(a_2) \longrightarrow a_1 = a_2$

$\frac{1}{a_1 - 4} + \cancel{1} = \frac{1}{a_2 - 4} + \cancel{1}$ $\Rightarrow a_1 - \cancel{4} = a_2 - \cancel{4}$

$\frac{1 + a_1 - 4}{a_1 - 4} = \frac{1 + a_2 - 4}{a_2 - 4}$ $a_1 = a_2$

$\frac{a_1 - 3}{a_1 - 4} = \frac{a_2 - 3}{a_2 - 4}$

3

$(a_2 - 3)(a_1 - 4) = (a_2 - 4)(a_1 - 3)$

$\cancel{a_1 a_2} - 4a_2 - 3a_1 \cancel{+12} = \cancel{a_1 a_2} - 3a_2 - 4a_1 \cancel{+12}$

$-4a_2 + 3a_2 = -4a_1 + 3a_1$ ✓

$-a_2 = -a_1$ $\therefore a_1 = a_2$ $\therefore$ it is one-to-one

(c) **[1 point]** The function f is not onto. What changes can you do to make it onto?

make the co-domain of the function f equal to the range (co-domain $= \{y \in \mathbb{R} \mid y \neq 1\}$) so that range = co-domain (onto)

1

(d) **[2 points]** Find $f \circ f$.

$f(f(x)) = f\left(\frac{1}{x-4} + 1\right) = f\left(\frac{x-3}{x-4}\right) = \frac{1}{\frac{x-3}{x-4} - 4} + 1$ ✓

2

$= \frac{1}{\frac{x-3-4(x-4)}{x-4}} + 1 = \frac{1}{\frac{-3x+13}{x-4}} + 1 = \frac{x-4}{-3x+13} + \frac{-3x+13}{-3x+13}$

$\frac{x-4-3x+13}{-3x+13} = \frac{-2x+9}{-3x+13}$

(2) Let $A = \{0, 1, 2, 3\}$. For each $a, b \in A$ define $a * b = |a - b|$. Answer all of the following questions from the table.

(a) **[3 points]** Fill in the operation table below for *

$0 * 0 = |0 - 0| = 0$
$1 * 0 = |1 - 0| = 1$

3

*	0	1	2	3
0	0	1	2	3
1	1	0	1	2
2	2	1	0	1
3	3	2	1	0

(b) **[1 point]** Is * closed under A? Why?

1

Yes, because every entry in the table $\in A$
$\forall_{a,b} \in A : a * b \in A$

(c) **[1 point]** Is * commutative under A? why ?

1

Yes, by checking the main diagonal. or by checking all
for every $a, b \in A$ $a * b = b * a$ symmetric

(d) **[1 point]** Give a counterexample to disprove that * is associative.

1

$a = 3$ $b = 1$ $c = 2$
$(a * b) * c = 0$ $\therefore (a*b) * c \neq a * (b * c)$
$a * (b * c) = 1$ $\therefore$ it is not associative

(e) **[1 point]** Which element is the identity element? (if exists) Why?

1

zero, because the row and column labeled by zero
is (0, 1, 2, 3) (the column and row labeled by identity element
must be $(a_1, a_2, a_3 \ldots a_n)$) ($a * e = a$, $e * a = a$)

(3) **[4 points]** Suppose G is a group. For any $a, b, c \in G$, prove that $(a * b * c)^{-1} = c^{-1} * b^{-1} * a^{-1}$.

4

	$(a * b * c) * (c^{-1} * b^{-1} * a^{-1})$	$(c^{-1} * b^{-1} * a^{-1}) * (a * b * c)$
$\because$ associative	$= a * (b * c * c^{-1} * b^{-1}) * a^{-1}$	$= c^{-1} * (b^{-1} * a^{-1} * a * b) * c$
$\because$ associative	$= a * b * (c * c^{-1}) * (b^{-1} * a^{-1})$	$= c^{-1} * b^{-1} * (a^{-1} * a) * b * c$
"	$= (a * b * e) * b^{-1} * a^{-1}$	$= (c^{-1} * b^{-1} * e) * b * c$
"	$= (a * b) * (b^{-1} * a^{-1})$	$= (c^{-1} * b^{-1}) * (b * c)$
"	$= a * (b * b^{-1}) * a^{-1}$	$= c^{-1} * (b^{-1} * b) * c$
"	$= (a * e) * a^{-1}$	$= c^{-1} * e * c$
	$= a * a^{-1} = e$	$= c^{-1} * c = e$

$\therefore c^{-1} * b^{-1} * a^{-1}$ is the inverse of $(a * b * c)$
$\therefore c^{-1} * b^{-1} * a^{-1} = (a * b * c)^{-1}$ #

(4) Answer the following questions. Each question is independent.

(a) **[3 points]** Write the general solution for a recurrence relation a_n with roots $-1, -1, -1, 3, 3, 6$.

3

$a_n = c_1(-1)^n + c_2 n(-1)^n + c_3 n^2(-1)^n + c_4(3)^n + c_5 n(3)^n + c_6(6)^n$

(b) **[3 points]** What is the characteristic equation for the recurrence relation

$$a_n = 2a_{n-1} + (-2)^n \cdot (n+2)^3 \text{ for all } n \geq$$

$a_n - 2a_{n-1} = (-2)^n \cdot (n+2)^3$

3

$(x-2)(x+2)^4 = 0$

(5) **[6 points]** Use the Substitution method to show that the solution for $a_n = a_{n-1} + c$, $n \geq 1$ is $a_n = a_0 + cn$, where c is a real constant.

$a_n = a_{n-1} + c$

$a_n = (a_{n-2} + c) + c = a_{n-2} + c + c = a_{n-2} + 2c$

$= (a_{n-3} + c) + c + c = a_{n-3} + c + c + c = a_{n-3} + 3c$

$\vdots$

at step k

$a_n = a_{n-k} + kc$

$\vdots$

until

$a_{n-k} = a_0$

$n - k = 0 \Rightarrow n = k$

6

$\therefore a_n = a_{n-n} + nc$

$a_n = a_0 + nc$ #

(6) **[8 points]** Use the Inhomogeneous approach to solve the following recurrence relation (without finding the constants).

$$T(1) = 1,\ \ T(n) = 4T(n/3) + n,\ \text{ for all } n = 3, 3^2, 3^3, 3^4 \ldots$$

$T(1) = 1$, $T(n) = 4T(\frac{n}{3}) + n$

Let $n = 3^k$, $k = 1, 2, 3, \ldots$

$T(3^k) = 4T(3^{k-1}) + 3^k$

$T(3^k) - 4T(3^{k-1}) = 3^k$

characteristic equation

$(x-4)(x-3) = 0$

the roots are

$r_1 = 4$ $r_2 = 3$

$T(3^k)$ $a_n = c_1(4)^k + c_2(3)^k$

$n = 3^k \Rightarrow \log_3 n = \log_3 3^k \Rightarrow \log_3 n = k \log_3 3$

$\Rightarrow \log_3 n = k$

8

$\therefore a_n = c_1(4)^{\log_3 n} + c_2(3)^{\log_3 n}$

$a_n = c_1 n^{\log_3 4} + c_2 n^{\log_3 3}$

$a_n = c_1 n^{\log_3 4} + c_2 n$